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The equivalence of R^*_δ-integral to D^*_δ-integral

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Abstract. Given two real numbers a and b where $a < b$. Let δ be a nonnegative real function on interval $[a, b]$. By changing the role of interval system on $\mathbf{R}$ by δ-fundamental interval system, we construct the δ-Henstock integral (R^*_δ-integral) of real function on $[a, b]$. We discuss some properties of the integral. Furthermore, we also discuss the equivalence of R^*_δ-integral to δ-special Denjoy integral (D^*_δ-integral).

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1. Introduction

There are two types on defining the integrable function in a cell: namely, descriptive type and constructive type. Descriptive type integral is defined by the characteristics of its primitive function, whilst the constructive integral is defined by the limit of Riemann sum of partition on the cell ([3]).

The examples of descriptive type integral are Lebesgue integral which is constructed by H. Lebesgue in 1968 ([7]), Approximately continuous Perron integral which is contructed by Burkill in 1931 ([2]) , Generalized Approximately Continuous integral has been introduced by Ganguly and Mukherjee ([4]), whilst Denjoy integral has been inruduced by Kubota ([9], [10], [11]), Restricted Denjoy integral ([5]), and δ-special Denjoy by

Manuharawati ([15]). The examples of constructive integral are Riemann-Lebesgue integral ([5]), Riemann Complete integral or Henstock integral which was constructed by R. Henstock in 1968 ([7]), Pettis integral ([16], [17]), and Generalized Riemann integral was constructed by McLeod ([8], [14]).

To accommodate the integral user, many mathematicians examine the relationship between the descriptive integral and constructive integral, and even researchers have proved the equivalence between the two types of integrals. These proofs have been done by Kubota in 1980, which proved that Henstock integral is equivalent to special Denjoy integral ([12]).

Bullen in 1980, has proved that the Burkill Approximately Continuous integral is equivalent to the Approximately Continuous Perron integral and the Pettis integral is equivalent to the Lebesgue integral ([1], [17]), and Gordon ([6]) has also proved that Generalized Riemann and Restricted integral are equivalent.

By substituting the δ-fine partition of $[a, b]$ in the definition of Henstock integral in $[a, b]$ with the δ-fundamental partition of $[a, b]$, we construct a Riemann type integral or constructive integral type here in which is called δ-Henstock or R^*_δ-integral and it is discussed some properties in it.

Furthermore, by using the ideas of Kubota ([12]), Bullen ([1]), Talagrand ([17]), Gordon ([6]), and using the concept of δ-continuity and its properties, it will be proved the equivalence of the R^*_δ-integral with D^*_δ-integral which constructed by Manuharawati ([15]).

2. Preliminaries

Given a positive function δ on a cell $[a, b]$. Let D_x be a δ-fundamental interval system of x and $\mathcal{G}$ be a collection of all D_x (Definition 2.1 in [15]).

If

$$\triangle = \{[l, m] : l, m \in D_x, l \leq x \leq m, D_x \in \mathcal{G}\}$$

then $\triangle$ is cover of $[a, b]$ and is called fundamental full cover (FFC) of $[a, b]$ and $\mathcal{G}$ is called a generator of FFC-$\triangle$.

Definition 2.1. Given a positive function δ on $[a, b]$, and $\mathcal{G} = \{D_x\}$ be a generator of FFC-$\triangle$ on $[a, b]$. A δ-fundamental partition on $[a, b]$ is a set

$$\mathcal{P} = \{a = a_0, a_1, a_2, , a_n = b; x_0, x_1, x_2, , x_n\} = \{[u, v]; x\}$$

with properties $a_{i-1} \leq x_i \leq a_i$ and $a_{i-1}, a_i \in D_{x_i} \in \mathcal{G}$.

Theorem 2.2 ([8, p.77] *For every FFC of a cell $[a, b]$ there exists a δ-fundamental partition of $[a, b]$.*

3. Results and discussion

Definition 3.1. Let δ be a nonnegative real function on a cell $[a, b]$. A function $f : [a, b] \to \mathbf{R}^*$ is said to be δ-Henstock integrable function on $[a, b]$, and denoted by $f \in R^*_\delta[a, b]$ if there exists a real number L such that for every real number $\varepsilon > 0$ there exists FFC-$\triangle$ on $[a, b]$ such that for any δ-fundamental partition $\mathcal{P} = \{[u, v]; x\}$ of $[a, b]$ implies

$$|L - (\mathcal{P}) \sum (f(x)(v - u))| < \varepsilon.$$

If $f \in R^*_\delta[a, b]$ then $f \in R^*_\delta[a, x]$ for every $x \in [a, b]$.

Furthermore, if $f \in R^*_\delta[a, b]$, there exists a function $F : [a, b] \to \mathbf{R}$ such that

$$F(x) = (R^*_\delta) \int_a^x f.$$

Consequently, Definition 3.1 is equivalent to the following definition.

Definition 3.2. Let δ be a nonnegative real function on a cell $[a, b]$. A function $f : [a, b] \to \mathbf{R}^*$ is said to be δ-Henstock integrable on $[a, b]$, if there exists a real function F on $[a, b]$ such that for every real number $\varepsilon > 0$ there exists a FFC-$\triangle$ on $[a, b]$ such that for any δ-fundamental partition $\mathcal{P} = \{[u, v]; x\}$ of $[a, b]$ implies

$$|(\mathcal{P})\sum(F(v-u) - f(x)(v-u))| = |F(a, b) - (\mathcal{P})\sum(f(x)(v-u))| < \varepsilon.$$

A function F such as is called a R^*_δ-primitive of f on $[a, b]$.

Theorem 3.3 (Henstock Lemma). *If $f \in R^*_\delta[a, b]$, which satisfies Definition 3.2, then for every subpartition $\mathcal{P}_*$ of $\mathcal{P}$ implies*

$$|(\mathcal{P}_*)\sum(F(u, v) - f(x)(v-u))| < 2\varepsilon.$$

Proof. Let ε be a real positive number. Since $f \in R^*_\delta[a, b]$, then there exists a δ-fundamental partition $\mathcal{P} = \{[u, v]; x\}$ of $[a, b]$ such that Definition 3.1 holds.

Let $\mathcal{P}_*$ be a subpartition of $\mathcal{P}$ and A is the union of all $[u, v]$ which $([u, v]; x) \in \mathcal{P}$ and $B = cl([a, b]) - A$.

It is easy to prove that $f \in R^*_\delta(B)$. So, if $\mathcal{P}_1 = \{[u, v]; x\}$ is a δ-fundamental partition of B then Definition 3.1 is hold if $\mathcal{P}$ is changed by $\mathcal{P}_1$.

If

$$\mathcal{P}_2 = \mathcal{P}_* \cup \mathcal{P}_1,$$

then $\mathcal{P}_2$ is a δ-fundamental partition of $[a, b]$. Consequently

$$\begin{aligned}&|(\mathcal{P}_*)\sum(F(u, v) - f(x)(v-u))| \leq |(\mathcal{P}_2)\sum(F(u, v) - f(x)(v-u))| \\ &+|(\mathcal{P}_1)\sum(F(u, v) - f(x)(v-u))| < \varepsilon + \varepsilon = 2\varepsilon.\end{aligned}$$

$\square$

Theorem 3.4. *If $f \in R^*_\delta[a, b]$ with a R^*_δ-primitive of f on $[a, b]$ is F then $F \in C_\delta[a, b]$.*

Proof. Let c be an arbitrary at $[a,b]$ and ε be a real positive number.

Choose

$$\gamma = \frac{\varepsilon}{2(b-a)[|f(c)|+1]}.$$

By applying Henstock's Lemma and axiom of δ-fundamental interval ([15, Axiom A_1]), there exist a δ-fundamental interval D_c such that

$$|F(v) - F(u)| < |f(c)|(v-u) + \frac{\varepsilon}{2} < \varepsilon.$$

If $u, v \in D_c$, $u \le c \le v$, $v - u < \gamma$. So, $F \in C_\delta[a,b]$. $\square$

Theorem 3.5. *If $f \in R^*_\delta[a,b]$ with a R^*_δ-primitive of f on $[a,b]$ is F then $F \in ACG^*_\delta[a,b]$.*

Proof. Let ε be an arbitrary real positive number. Since $f \in R^*_\delta[a,b]$, then there is a monotone function φ on $[a,b]$ such that $\varphi(a,b) < \varepsilon$ and for every $x \in [a,b]$ there is a δ-fundamental interval D_x such that for every

$$u, v \in D_x \bigcap [a,b]$$

and $u \le x \le v$, we have

$$|F(v) - F(u) - f(x)(v-u)| \le \varphi(u,v)$$

or

$$|F(v) - F(u)| \le |f(x)|(v-u) + \varphi(u,v)$$

If for every $i, n \in \mathbf{N}$, we define

$$\begin{aligned} Y_n &= \{x \in [a,b] : |f(x) \le n\} \\ Y_{n,i} &= Y_n \cap [a + \frac{i-1}{n}, a + \frac{i}{n}], \end{aligned}$$

then $[a,b] = \cup_{i,n\in\mathbf{N}} Y_{n,i}$.

Let x be an arbitrary element of Y_n and

$$D'_x = D_x \cap [x - \frac{1}{n}, x + \frac{1}{n}],$$

then by A_1 [15], D'_x is a δ-fundamental interval at x.

Let (x_k) be an arbitrary sequence on $Y_{n,i}$, then there is a sequence of non overlapping fundamental δ-interval (D_{x_k}).

Since $F \in C_\delta[a,b]$, then for any $u_k \in Y_{n,i} \cap D_{x_k}$, and

$$\begin{aligned} a_k &= min\{x_k, u_k\}, \\ b_k &= max\{x_k, u_k\}, \end{aligned}$$

we have $F \in C_\delta[a_k, b_k]$.

So, there are real numbers α_k, β_k such that

$$\omega(F;[a_k,b_k]) = |F(\alpha_k) - F(\beta_k)|.$$

Consequently, we have

$$\begin{aligned} \sum_{k\in\mathbf{N}} \omega(F;[a_k,b_k]) &= \sum_{k\in\mathbf{N}} |F(\alpha_k) - F(\beta_k)| \\ &\leq \sum_{k\in\mathbf{N}} |f(\alpha_k)||\alpha_k - \beta_k| + \sum_{k\in\mathbf{N}} |\varphi(\alpha_k) - \varphi(\beta_k)| \\ &< n \sum_{k\in\mathbf{N}} |\alpha_k - \beta_k| + \varepsilon. \end{aligned}$$

If $\displaystyle\sum_{k\in\mathbf{N}} |\alpha_k - \beta_k| < \frac{\varepsilon}{n} = \gamma$, then

$$\sum_{k\in\mathbf{N}} \omega(F;[a_k,b_k]) < 2\varepsilon.$$

It means that, $F \in AC^*_\delta(Y_{n,i})$.

Consequently, $F \in ACG^*_\delta([a,b])$. $\square$

Theorem 3.6. *A function f is R^*_δ-integrable on $[a,b]$ if only if f is D^*_δ-integrable on $[a,b]$.*

Proof. Sufficient condition.

Let F be a R_δ^*-primitive of f on $[a,b]$. It is easy to prove that $D_\delta F(x) = f(x)$ a.e on $[a,b]$.

By Theorem 3.4 and Theorem 3.5, we have $F \in C_\delta[a,b]$ and $F \in ACG_\delta^*([a,b])$.

By applying [15, Definition 3.1], we have $f \in D_\delta^*([a,b])$.

Necessary condition. Since $f \in D_\delta^*([a,b])$, then we have a real function F defined on $[a,b]$ such that

$$D_\delta F(x) = f(x) \text{ a.e } \text{ on } [a,b]$$

and

$$F \in C_\delta[a,b] \bigcap ACG_\delta^*([a,b]) \text{ ([15, Definition 3.1])}.$$

If

$$Y = \{x \in [a,b] : D_\delta F(x)$$

exists but

$$D_\delta F(x) \neq f(x).v.D_\delta F(x)$$

does not exist, then $\mu(Y) = 0$.

Let ε be an arbitrary positive real number and $x \in [a,b]$.

There are two cases of x, i.e. $x \in [a,b] - Y$ or $x \in Y$.

Case 1. If $x \in [a,b] - Y$, then there exists a δ-fundamental D'_x such that

$$|F(v) - F(u) - f(x)(v-u)| < \frac{\varepsilon(v-u)}{b-a}$$

for any $u \leq x \leq v$, $u, v \in D'_x \bigcap [a,b]$.

Case 2. If $x \in Y$, then for any $n \in \mathbf{N}$ we have constructed a set

$$Y_n = \{x \in [a,b] : |f(x)| \leq n\}.$$

Consequently, we have

$$Y = \bigcup_{n \in \mathbf{N}} Y_n.$$

Since $F \in ACG^*_\delta[a,b]$ then there is a sequence of set (X_i) such that

$$[a,b] = \cup_{i \in \mathbf{N}} X_i$$

and

$$F \in AC^*_\delta(X_i)$$

for every $i \in \mathbf{N}$. If for every $n, m \in \mathbf{N}$,

$$S_{n,m} = Y_n \cap X_m$$

then we have:

(i) $\bigcup_{n \in \mathbf{N}} \bigcup_{m \in \mathbf{N}} S_{n,m} = Y$,

(ii) for every $x \in S_{n,m}$, $|f(x)| \leq 2^n$,

(iii) $F \in AC^*_\delta(S_{n,m})$.

Since $F \in AC^*_\delta(S_{n,m})$, then there exists a positive real number $\gamma_{n,m}$ such that for any sequence (x_i) on $S_{n,m}$ there exists a sequence of non-overlapping δ-fundamental interval (D''_{x_i}) such that for any

$$u_i \in D''_{x_i} \bigcap S_{n,m}$$

and

$$\sum_{i \in \mathbf{N}} |x_i - u_i| < \gamma_{n,m}$$

we have

$$\sum_{i \in \mathbf{N}} \omega(F; [a_i, b_i]) < \frac{\varepsilon}{2^{n+m}}$$

if $a_i = min\{x_i, u_i\}$ and $b_i = max\{x_i, u_i\}$.

Since $S_{n,m} \subset Y$, then $\mu(S_{n,m}) = 0$. So, there is a sequence of open interval $(I^j_{n,m} = (u,v))$, $u \leq x \leq v$ such that

$$S_{n,m} \subset \bigcup_{j \in \mathbf{N}} I^j_{n,m}$$

and

$$\sum_{j\in\mathbf{N}} \mu(I^j_{n,m}) < \frac{\varepsilon}{2^{2n+m}}.$$

If $x \in Y$ and $x \in I^k_{n,m}$ for a some k, then we have chosen a δ-fundamental interval $D_x{'''}$ such that $D_x{'''} \subset I^k_{n,m}$.

If $x \in Y$ and x is end point of $I^k_{n,m}$, then there is not δ-fundamental interval D_x such that

$$D_x \subset I^k_{n,m}.$$

Let

$$A = \{x \in Y \text{ and } x \text{ is end point } of \ I^k_{n,m}\}.$$

Then A is a countable set.

Since $F \in C_\delta[a,b]$ then for every $x \in A$ there exists a δ-fundamental interval D^*_x such that for any $t \in D^*_x \cap [a,b]$ implies

$$|F(t) - F(x)| < \frac{\varepsilon}{2^{k+2}}.$$

Consequently, if $u, v \in D^*_x \bigcap [a,b]$ with $u \le x \le v$ we have

$$|F(v) - F(u)| \le |F(v) - F(x)| + |F(x) - F(u)| < \frac{\varepsilon}{2^{k+1}}.$$

Let $\triangle$ be a FFC on $[a,b]$ with generator $\mathcal{G} = \{D_x\}$,

$$D_x = \begin{cases} D'_x & ;\ \mathrm{if\ x \in [a,b] - Y} \\ D''_x \cap D_x{'''} & ;\ \mathrm{if\ x \in Y \cap I^k_{n,m}} \\ D^*_x & ;\ \mathrm{if\ x \in A.} \end{cases}$$

If $P = \{[u,v];x\}$ is an arbitrary δ-fundamental partition of $[a,b]$ relative to $\triangle$, then

$$\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$$

with

$$\begin{aligned} \mathcal{P}_1 &= \{[u,v];x\} \subset \mathcal{P},\ x \in [a,b] - Y; \\ \mathcal{P}_2 &= \{[u,v];x\} \subset \mathcal{P},\ x \in [a,b] \cap I^k_{n,m};\ \ and \\ \mathcal{P}_3 &= \{[u,v];x\} \subset \mathcal{P},\ x \in A. \end{aligned}$$

Consequently,

$$\begin{aligned} & |(\mathcal{P})\sum(F(v)-F(u)-f(x)(v-u))| \\ \leq\ & |(\mathcal{P}_1)\sum(F(v)-F(u)-f(x)(v-u))| \\ +\ & |(\mathcal{P}_2)\sum(F(v)-F(u)-f(x)(v-u))| \\ +\ & (\mathcal{P}_3)\sum(F(v)-F(u)-f(x)(v-u))| < 4\varepsilon. \end{aligned}$$

This means that $f \in R^*_\delta[a,b]$. □

3. Conclusion

By changing the concept of interval system by δ-fundamental interval system over $\mathbf{R}$, it is successfully constructed δ-Henstock integral (R^*_δ-integral). Furthermore, we had proved that this integral is equvalen to δ-special Denjoy integral (D^*_δ-integral).

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References

[1] P.S. Bullen, *The Burkill Approximately Continuous Integral*, J. Aust. Math. Soc., Ser. A., 35 (1983), 236-53.

[2] J.C. Burkill, *The approximately continuous Perron integral*, Math. Zeit, 34 (1931), 270-278.

[3] I. Fleischer, *The Generalized Reimann Integral*, Windsor Mathematics Statistics Report, WMSR 92, 1992.

[4] D.K. Ganguly and R. Mukherjee, *Two important extension theorems for the GAP-integral*, J. Indones. Math. Soc. 17 (2011), 49-58.

[5] R.A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron and Henstock*, Amer. Math. Soc., 4 (1994).

[6] R. Gordon. *Equivalence of the Generalized Riemann an Restricted Denjoy integrals*, Real Analysis Exchange, 12 (1986/1987), 551-574.

[7] R. Henstock, *The Riemann-type integral of Lebesgue power*, Canad. J. Math., 20 (1968), 79-87.

[8] S. Hu and V. Lakshmikantham, *Some Remarks on Generalized Riemann Integral*, Journal of Mathematical Analysis and Applications, 137 (1989), 515-527.

[9] Y. Kubota, *An Integral of the Denjoy Type I*, Proc. Japan Acad., 40 (1964), 713-717.

[10] Y. Kubota, *An Integral of the Denjoy Type II*, Proc. Japan Acad., 42 (1966), 737-742

[11] Y. Kubota, *An Integral of the Denjoy Type III*. Proc. Japan Acad., 43 (1967), 949-952.

[12] Y. Kubota, *An elementary theory of the special Denjoy integral*, Math. Japonica, 24 (1980), 507-520.

[13] Y. Kubota, *A direct proof that the RC-integral is equivalent to the D^*-integral*, Proc. of the American Math. Soc., 80 (1980), 293-296.

[14] R.M. McLeod, *The generalized Reimann integral*, Carus Math 20, MAA 1980.

[15] Manuharawati, *Some Convergence Theorems on δ-Denjoy Integral*, Adv. Sci. Lett., 23 (2017), 11991-11994.

[16] K. Naralenkov, *On Denjoy type extensions of the Pettis integral*, Czechoslovak Mathematical Journal, 60 (2010), 737-750.

[17] M. Talagrand, *Pettis Integral and Measure Theory*, Mem. Ame. Math. Soc., 307 (1984), 56-63.

[18] B.S. Thomson, *On full covering properties*, Real Analysis Exchange, 6 (1980/1981), 77-93.

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